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1994 J. Phys. A: Math. Gen. 27 2037

(http://iopscience.iop.org/0305-4470/27/6/026)

J. Phys. A: Math. Gen. 27 (1994) 2037–2043. Printed in the UK

An analogue of the unitary displacement operator for the q-oscillator

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Received 25 November 1993

Abstract. Using the concept of a non-standard Hilbert space over the quantum complex plane recently introduced by Kowalski *et al* [1], we construct a unitary q-analogue of the Weyl-displacement operator. We investigate the q-displaced vacuum states and show that they exhibit properties analogous to coherent states in the undeformed theory. They are, however, distinct from any q-coherent states previously found in the literature.

Coherent state techniques have been among the standard tools of theoretical physics for many years and in that time they have undergone extensive development. Although originally proposed by Schrödinger [2] in 1926, their properties were really first investigated by Glauber [3–5] in the context of quantum optics. Subsequent work by such authors as Klauder [6], Solomon [7], Perelomov [8, 9], Rasetti [10, 11] and Gilmore [12] has extended their physical application as well as generalizing the concept to coherent states of arbitrary Lie groups. An extensive bibliography of such work is provided by Klauder and Skagerstam [13].

In one of his original papers [5], Glauber gave three approaches to the coherent states of the conventional boson operators (i.e. the generators of the Heisenberg–Weyl algebra). Coherent states could be defined as:

(i) states which minimize the field quadrature uncertainty product, i.e. minimum uncertainty states;

(ii) states which are the normalized eigenstates of the lowering operator of the algebra, i.e. annihilation operator eigenstates; and

(iii) states which are formed by the action of a unitary displacement operator on a lowest weight (vacuum) vector, i.e. displaced vacuum states.

In the case of the Heisenberg–Weyl algebra, these three definitions are shown to be equivalent. However, if we allow the eigenvalues of the states to belong not to a field but to some other algebraic structure, care needs to be taken to distinguish between the different approaches. If we consider the annihilation operator eigenstates and the displaced vacuum states, we see that if the 'eigenvalue' of the eigenstate does not commute with the displacement operator, then in this extension of the theory we need to introduce the concepts of left and right eigenvalues.

Recently there has been much interest in the subject of deformations of algebraic structures and quantum groups [14–16] in particular. This has led many authors to undertake an investigation of the properties of q-deformations of the ordinary boson algebra. The q-bosons studied fall naturally into two types (although there are links between them, see [17, 18]). One type, first discovered by Arik and Coon [19], uses the basic numbers of

0305-4470/94/062037+07$19.50

classical q-analysis and can be termed *maths*-type q-bosons. The states associated with these are related to the extensive mathematics literature of q-special functions [20, 21], which is an additional reason for our use of this term. The second type proposed by Macfarlane [22] and Biedenharn [23] in connection with the representation theory of quantum groups and arousing much interest in the physics literature can be termed *physics*-type q-bosons. The coherent states of both types of q-boson have been extensively studied [24–26]. Most authors have tried to find coherent states for the q-boson algebra in terms of normalized eigenstates of the deformed annihilation operator. While much work has been done, it has not proved possible to give a displacement operator which produces such a state from the vacuum vector, although Jannussis [27] and, more recently, Zhedanov [28] (in the context of his (u, v)-algebras) have proposed operators which have some of the properties required. Unfortunately, the relation between states produced from these operators and the normalized eigenstates of the deformed annihilation operator remains problematic.

In almost all cases the base field has been assumed to be the complex plane (although quantum plane variables have been considered by Fairlie and others in [29–31]). In a recent paper, however, Kowalski and Rembielinski [1] introduced a new generalization of the conventional theory applicable to the *maths*-type q-bosons. Instead of using $\mathbb{C}$ as the base field, they use a deformation of the complex plane, $\mathbb{C}_q$ [32, 33]. Formally $\mathbb{C}_q$ is the algebra $\mathbb{C}[z, z^*]/B_q$, i.e. the quotient of the involutive algebra freely generated by z and z^*, and B_q, the bi-ideal determined by the re-ordering rule

$$zz^* = qz^*z \tag{1}$$

where q is a non-zero real parameter.

$\mathbb{C}_q$ is also a left $U_q(2)$-module where $U_q(2)$ is the two-dimensional unitary matrix quantum group of Manin [15] and Woronowicz [16].

Kowalski and Rembielinski showed that, analogously to conventional quantum mechanics where the Hilbert space of states of the oscillator is a left $\mathbb{C}$-module, it is possible to define the Hilbert space of states of the q-deformed oscillator as a left $\mathbb{C}_q$-module. This allows an extension of the concept of conventional coherent states (over $\mathbb{C}$) to states over $\mathbb{C}_q$.

In this paper, we propose an analogue of the unitary displacement operator of conventional coherent state theory. Working within the framework outlined in [1], we define displaced vacuum states and show that these are analogues of the eigenstates of the deformed annihilation operator. Such states are related to those given by Kowalski and Rembielinski by a similarity transformation. Moreover, it is easy to show that the displaced vacuum states have the same quantum noise dispersion value as the undisplaced vacuum state (given conventional definitions of the field quadratures in terms of creation and annihilation operators). This should be contrasted with the states given in [1].

We consider the q-oscillator specified by

$$aa^\dagger - qa^\dagger a = 1 \tag{2}$$

together with the algebra $\mathbb{C}_q$ with generators z and z^* $(= z^\dagger)$ which are assumed to commute with a and $a^\dagger$.

We use the Jackson q-exponential function [34]

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{3}$$

where $[n] = (1-q^n)/(1-q)$ is the basic number of q-analysis [21].

It is known [35] that

$$E_q(X)YE_{q^{-1}}(-X) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!}[X,Y]_n \tag{4}$$

where $[X,Y]_0 = Y$ and

$$[X,Y]_{n+1} = X[X,Y]_n - q^{n-1}[X,Y]_n X \tag{5}$$

and from this it is straightforward to show that

$$E_q(za^\dagger)z^*aE_{q^{-1}}(-za^\dagger) = z^*a - z^*z \tag{6}$$

which implies

$$E_q(za^\dagger)E_q(z^*a)E_{q^{-1}}(-za^\dagger) = E_q(z^*a - z^*z). \tag{7}$$

Using the results [35, 21]

$$E_q(X)E_{q^{-1}}(-X) = 1 \tag{8}$$

and [36, 37]

$$E_q(X)E_q(Y) = E_q(X+Y) \qquad \text{if } YX = qXY \tag{9}$$

we obtain the following analogue of the conventional reordering property

$$E_q(za^\dagger)E_q(z^*a) = E_q(z^*a)E_q(-z^*z)E_q(za^\dagger). \tag{10}$$

Taking the inverse of (10) and letting $z \to -z$, $z^* \to -z^*$, we obtain

$$E_{q^{-1}}(z^*a)E_{q^{-1}}(za^\dagger) = E_{q^{-1}}(za^\dagger)E_{q^{-1}}(z^*z)E_{q^{-1}}(z^*a). \tag{11}$$

Equations (10) and (11) allow us to reorder products of q-exponentials (see [28] and [30] for other examples of this procedure).

Consider the operator $U(z,z^*)$ defined by

$$U(z,z^*) = E_q(-z^*z)^{1/2}E_q(za^\dagger)E_{q^{-1}}(-z^*a). \tag{12}$$

The Hermitian conjugate of this is

$$U(z,z^*)^\dagger = E_{q^{-1}}(-za^\dagger)E_q(z^*a)E_q(-z^*z)^{1/2}. \tag{13}$$

Using (10) and (11), it can be shown that U is a unitary operator i.e. that

$$U(z,z^*)^\dagger = U(z,z^*)^{-1}. \tag{14}$$

Furthermore, in the limit $q \to 1$, $U(z,z^*) \to D(z)$, where $D(z)$ is the Weyl-displacement operator of conventional coherent state theory.

$$D(z) = \exp(-\tfrac{1}{2}|z|^2)\exp(za^\dagger)\exp(-z^*a) = \exp(za^\dagger - z^*a). \tag{15}$$

To show that $U(z,z^*)$ is a q-deformed displacement operator, we make use of the following result:

$$a(a^\dagger)^n - q^n(a^\dagger)^n a = [n]_q(a^\dagger)^{n-1} \tag{16}$$

which, since $[n]_{q^{-1}} = q^{1-n}[n]_q$, leads to the formula

$$E_{q^{-1}}(-za^\dagger)a - aE_{q^{-1}}(-zq^{-1}a^\dagger) = zq^{-1}E_{q^{-1}}(-za^\dagger). \tag{17}$$

Now consider the operator product $U(z,z^*)^\dagger aU(z,z^*)$

$$U(z,z^*)^\dagger aU(z,z^*) = E_{q^{-1}}(-za^\dagger)E_q(z^*a)E_q(-z^*z)^{1/2}aE_q(-z^*z)^{1/2}E_q(za^\dagger)E_{q^{-1}}(-z^*a) \tag{18}$$

$$\begin{aligned} &= \{aE_{q^{-1}}(-zq^{-1}a^\dagger) + zq^{-1}E_{q^{-1}}(-za^\dagger)\}E_q(z^*a)E_q(-z^*z)E_q(za^\dagger)E_{q^{-1}}(-z^*a) \\ &= aE_{q^{-1}}(-zq^{-1}a^\dagger)E_q(za^\dagger) + zq^{-1} \\ &= aE_q(za^\dagger)E_q(zq^{-1}a^\dagger)^{-1} + zq^{-1}. \end{aligned} \tag{19}$$

From the definition of the derivative of the q-exponential, it can be seen that

$$E_q(za^\dagger)E_q(zq^{-1}a^\dagger)^{-1} = 1 + (q-1)q^{-1}za^\dagger \tag{20}$$

so

$$U(z,z^*)^\dagger aU(z,z^*) = a\{1 + (q-1)q^{-1}za^\dagger\} + zq^{-1}. \tag{21}$$

Using $aa^\dagger = [N+1]$, which follows from (2), we obtain the result

$$U(z,z^*)^\dagger aU(z,z^*) = a + zq^N. \tag{22}$$

Since $q^N = [a, a^\dagger]$, we can also write this as

$$U(z,z^*)^\dagger aU(z,z^*) = a + z[a, a^\dagger]. \tag{23}$$

We therefore have an analogue of the conventional ($q = 1$) formula

$$D(z)^\dagger aD(z) = a + z[a, a^\dagger] = a + z1 \tag{24}$$

with $D(z)$ defined as in (15).

Because of the unitarity of $U(z,z^*)$, the transformation

$$a \to a' = U(z,z^*)^\dagger aU(z,z^*) \tag{25}$$

is a non-trivial automorphism of the q-boson algebra (2).

If we now define a set of displaced vacuum states by

$$|z,z^*) = U(z,z^*)|0) \tag{26}$$

then by virtue of the unitarity of U and (22),

$$aU(z,z^*) = U(z,z^*)\{a + zq^N\}. \tag{27}$$

Therefore

$$aU(z, z^*)|0\rangle = U(z, z^*)\{a + zq^N\}|0\rangle \tag{28}$$
$$= U(z, z^*)z|0\rangle \tag{29}$$
$$= U(z, z^*)|0\rangle z \tag{30}$$

i.e. $|z, z^*\rangle$ is a right 'eigenstate' of a with right 'eigenvalue' z,

$$a|z, z^*\rangle = |z, z^*\rangle z. \tag{31}$$

In the $q \to 1$ limit where z, z^* commute and a, $a^\dagger$ are just the usual bosonic operators, this becomes the familiar statement that displaced vacuum states are eigenstates of the annihilation operator.

$$a|z\rangle = z|z\rangle. \tag{32}$$

Since $|z, z^*\rangle = U(z, z^*)|0\rangle$, we can express these states in terms of the usual q-boson number states $|n\rangle$.

$$|z, z^*\rangle = E_q(-z^*z)^{1/2} E_q(za^\dagger) E_{q^{-1}}(-z^*a)|0\rangle \tag{33}$$
$$= E_q(-z^*z)^{1/2} \sum_{n=0}^{\infty} \frac{z^n (a^\dagger)^n}{[n]_q!}|0\rangle \tag{34}$$
$$= E_q(-z^*z)^{1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}}|n\rangle \tag{35}$$

so

$$\langle n|z, z^*\rangle = E_q(-z^*z)^{1/2} \frac{z^n}{\sqrt{[n]_q!}}. \tag{36}$$

In [1], Kowalski and Rembielinski defined another set of normalized states $|z, z^*)$ by

$$|z, z^*) = E_q(za^\dagger) E_q(-z^*z)|0\rangle \tag{37}$$

which have the property that $|z, z^*)$ are eigenstates of the annihilation operator with left eigenvalue z,

$$a|z, z^*) = z|z, z^*). \tag{38}$$

These eigenstates of the annihilation operator are related to the displaced vacuum states $|z, z^*\rangle$ by the similarity transformation

$$|z, z^*) = E_{q^{-1}}(z^*z)^{1/2} |z, z^*\rangle E_q(-z^*z)^{1/2}. \tag{39}$$

This illustrates that if $q \neq 1$, the two conventional definitions of coherent states as either eigenstates of the annihilation operator or displaced vacuum states are not equivalent for the type of q-deformed system constructed here.

To consider the quantum noise dispersion value in the displaced vacuum states, we use the (deformed) field components x and p defined by

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad \text{and} \qquad p = \frac{1}{\mathrm{i}\sqrt{2}}(a - a^\dagger). \tag{40}$$

The deformed commutation relation for a and $a^\dagger$ leads to the following dispersion relation

$$(\Delta x)_z(\Delta p)_z = (\Delta x)_0(\Delta p)_0 = \tfrac{1}{2} \tag{41}$$

where $(\Delta)_z^2$ and $(\Delta)_0^2$ are the quadrature variances in the displaced and undisplaced vacuum states respectively.

Therefore, just as in the conventional case [5], both the undisplaced and displaced vacuum states have the same value for the quantum noise dispersion.

It is interesting to note that we can define another unitary operator $V(z, z^*)$ by

$$V(z, z^*) = U(-z, -z^*)^\dagger \tag{42}$$

which leads to the alternative shift automorphism

$$a' = V(z, z^*) a V(z, z^*)^\dagger \tag{43}$$

$$= a - zq^N. \tag{44}$$

This can be used to define states which are eigenstates of a new deformed annihilation operator b, with left eigenvalue z, where b obeys the boson equation

$$bb^\dagger - q^{-1}b^\dagger b = q^{-2N}. \tag{45}$$

We conclude by making a few remarks concerning possible extensions of the theory of coherent states over non-commuting algebras rather than fields. It is known that a consistent extension of the theory of q-coherent states to q-squeezed states has not yet been successful. However, it has been pointed out by one of the authors [38] that the formalism discussed here might be an appropriate vehicle for developing the theory. In conventional quantum optics (i.e. with undeformed creation and annihilation operators A and $A^\dagger$), it is known that it is possible to define a squeezed state $|\xi, \lambda\rangle$ as a normalized state for which

$$(A + \xi A^\dagger)|\xi, \lambda\rangle = \lambda|\xi, \lambda\rangle \tag{46}$$

which gives

$$|\xi, \lambda\rangle = \mathcal{N}^{-1} \exp(-\tfrac{1}{2}\xi(A^\dagger)^2) \exp(\lambda A^\dagger)|0\rangle \tag{47}$$

where $\mathcal{N}$ is the normalization constant.

In [38] it was shown that for *maths*-type q-bosons, if we define a state $|\xi, \lambda\rangle$ by

$$|\xi, \lambda\rangle = \mathcal{N}_q^{-1} E_{q^2}(-(1/[2])\xi(a^\dagger)^2) E_q(\lambda a^\dagger)|0\rangle \tag{48}$$

then once again,

$$(a + \xi a^\dagger)|\xi, \lambda\rangle = \lambda|\xi, \lambda\rangle \tag{49}$$

provided $\lambda\xi = q^2\xi\lambda$.

This suggests that a generalization of $\mathbb{C}_q$ might be appropriate as the base algebra for constructing both q-squeezed states and a q-analogue of the unitary squeezing operator.

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